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Symmetry Properties of Scattering Amplitudes and Applications to Inverse Problems

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For the transformations of the potential $q \to q \circ L$, where L is one of the following transformations: rotation, scaling, or translation, the corresponding transformation of the scattering amplitude is given. A necessary and sufficient condition on the scattering amplitude for the scatterer to be spherically symmetric is derived as an application of the above results. © 1991 Academic Press, Inc.

I. INTRODUCTION

Let

$$[\nabla^2 + k^2 - q(x)]\, u = 0 \qquad \text{in} \quad \mathbb{R}^3,\ k > 0, \tag{1}$$

$$u = u_0(x, \theta, k) + A_q(\theta', \theta, k)\, r^{-1} \exp(ikr) + o(r^{-1}),$$

$$r = |x| \to \infty,\ xr^{-1} = \theta'$$

$\theta, \theta' \in S^2$, the unit sphere, $u_0(x, \theta, k) := \exp(ik\theta \cdot x)$.

We are interested in the following question: if one changes $q(x)$ in (1) to $q \circ L$, what is the corresponding change in $A_q(\theta', \theta, k)$? For example, let

$$(q \circ L)(x) = (q \circ R)(x) := q(R^{-1}x), \qquad R \in O(3), \tag{2}$$

where $O(3)$ is the group of rotations in $\mathbb{R}^3$, or

$$(q \circ L)(x) = (q \circ \lambda)(x) := q(\lambda x), \qquad \lambda > 0 \tag{3}$$

or

$$(q \circ L)(x) = (q \circ T_{-a})(x) := q(x - a), \qquad a \in \mathbb{R}^3. \tag{4}$$

Here R, λ, and T_{-a} are operators of rotation, scaling by a factor $\lambda > 0$ and translation by a vector $-a$, respectively.

409/156/2-3

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The basic results are the following formulas for the corresponding transformation of the scattering amplitudes:

$$A_q(\theta', \theta) = A_{q \circ R}(R\theta', R\theta), \qquad (q \circ R)(x) := q(R^{-1}x) \tag{5}$$

$$A_q(\theta', \theta, k) = \lambda A_{\lambda^2 q \circ \lambda}(\theta', \theta, \lambda k), \qquad (\lambda^2 q \circ \lambda)(x) = \lambda^2 q(\lambda x) \tag{6}$$

$$A_q(\theta', \theta, k) = A_{q \circ T_{-a}}(\theta', \theta, k) \exp\{ik(\theta' - \theta) \cdot a\}, \quad (q \circ T_{-a})(x) = q(x-a). \tag{7}$$

As an application, we prove that a necessary and sufficient condition for $q(x)$ to be spherically symmetric is that

$$A(\theta', \theta, k) = A(\theta' \cdot \theta, k), \qquad \forall \theta', \theta \in S^2 \tag{8}$$

at a fixed $k > 0$ if

$$q \in Q_b := \{q : q = \bar{q},\ q \in L^2(B_b),\ q = 0 \text{ for } |x| > b\}, \tag{9}$$

where $B_b := \{x : x \in \mathbb{R}^3, |x| \leqslant b\}$, $b > 0$ is an arbitrary fixed number and the overbar denotes complex conjugate. A similar result is proved for scattering by an obstacle. Namely, let

$$(\nabla^2 + k^2)u = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^3,\ k > 0 \tag{10}$$

$$u_N + \zeta(s)u = 0 \qquad \text{on} \quad \Gamma,\ \operatorname{Im} \zeta(s) \geqslant 0 \tag{11}$$

$$u \text{ is of the form (2) with } A_{\Gamma,\zeta}(\theta', \theta, k) \text{ in place of } A_q(\theta', \theta, k). \tag{12}$$

Here Ω is the exterior of a bounded domain D with a smooth connected boundary Γ, N is the unit normal to Γ pointing into Ω.

Let us formulate the results (see also [10] and [11]).

THEOREM 1. *Formulas* (5)–(7) *hold.*

THEOREM 2. *Assume that* (8) *holds and* $A(\theta', \theta, k) := A_q(\theta', \theta, k)$. *Then* (8) *holds at a fixed* $k > 0$ *iff* $q(x) = q(r)$.

THEOREM 3. *Assume that* (8) *holds at a fixed* $k > 0$ *with* $A(\theta', \theta, k) := A_{\Gamma,\zeta}(\theta', \theta, k)$. *Then* Γ *is a sphere and* $\zeta(s) = const$. *The converse is trivially true.*

In Section II we give proofs. The results in Theorems 2 and 3 are based on formula (5) and the uniqueness theorems established in [1–4]. The ideas from [5, 6] are used. Our results cover the result in [7], where the two-dimensional scattering by obstacle with Dirichlet boundary condition has been studied, and the results in [8], where the scattering by an obstacle with Dirichlet and Neumann boundary conditions has been treated and the scattering by a locally inhomogeneous acoustic medium has been treated. In the last case in [8] condition (8) is assumed to hold for all $k>0$, while our Theorem 2 covers this case with the weaker assumption that (8) holds at a single $k>0$. *It follows from our result that if* (8) *holds at a single* $k>0$ *then* (8) *holds for all* $k>0$ (*provided that* (9) *holds*).

II. Proofs

1. *Proof of Formulas* (5)–(7)

Formulas (5)–(7) are direct consequences of the definition of the scattering amplitude.

Formula (5) means that the scattering amplitude is the same in a coordinate system τ and the coordinate system $R\tau$ in which each vector becomes Ra, where $R \in O(3)$ is an arbitrary rotation, and $O(3)$ is the group of rotations in $\mathbb{R}^3$. The "rotated" potential $q \circ R$ is defined in (5).

A formal derivation of formulas (5)–(7) is based on writing the asymptotics of the scattering solution $u(x, \theta, k)$ in the new coordinate system. This derivation is the same in all three cases. Let us give it briefly.

(a) *Formula* (5). Consider

$$[\nabla_x^2 + k^2 - q(R^{-1}x)]\, u = 0 \qquad \text{in } \mathbb{R}^3 \tag{13}$$

$$u(x, \theta, k) = \exp(ik\theta \cdot x) + A_{q \circ R}(\theta', \theta, k)|x|^{-1} \times \exp(ik|x|) + o(|x|^{-1}), \qquad \theta' = \frac{x}{|x|} := x^0. \tag{14}$$

Let $\xi := R^{-1}x$, $\xi^0 := \xi/|\xi| = R^{-1}\theta'$, $|\xi| = |x|$. Note that

$$R\theta' \cdot R\theta = \theta' \cdot \theta, \qquad Rx \cdot a = x \cdot R^{-1}a \tag{14'}$$

since $R' = R^{-1}$ for any $R \in O(3)$. Here R' is the transposed operator of rotation. Note that $\nabla_x^2 = \nabla_\xi^2$ is invariant under rotations. Write (13) and (14) in the ξ-coordinates:

$$[\nabla_\xi^2 + k^2 - q(\xi)]\, w = 0, \tag{15}$$

$$u = \exp(ik\theta \cdot R\xi) + A_{q \circ R}(\theta', \theta, k)|\xi|^{-1} \times \exp(ik|\xi|) + o(|\xi|^{-1}), \qquad \theta' = R\xi^0. \tag{16}$$

Note that, by (14), $\theta \cdot R\xi = R^{-1}\theta \cdot \xi$. Thus, the function (16) is the scattering solution

$$w = \exp(ikR^{-1}\theta \cdot \xi) + A_q(\alpha', R^{-1}\theta)|\xi|^{-1} \exp(ik|\xi|) + o(|\xi|^{-1}) \quad (16')$$

corresponding to the incident direction $R^{-1}\theta$. Let $R^{-1}\theta := \alpha$, $R^{-1}\theta' = \alpha' := \xi^0$. Then

$$A_q(\alpha', \alpha) = A_{q \circ R}(R\alpha', R\alpha), \qquad \forall \alpha, \alpha' \in S^2.$$

This is formula (5).

(b) *Formula* (6). Consider

$$[\nabla_x^2 + k^2 - q(\lambda x)]\, u = 0 \qquad \text{in } \mathbb{R}^3, \tag{17}$$

$$u = \exp(ik\theta \cdot x) + A_{q \circ \lambda}(\theta', \theta, k)|x|^{-1} \exp(ik|x|) + o(|x|^{-1}). \tag{18}$$

Let $\xi = \lambda x$. Then (17) can be written as

$$\left[\nabla_\xi^2 + \frac{k^2}{\lambda^2} - \frac{q(\xi)}{\lambda^2}\right] w = 0. \tag{19}$$

The scattering solution, corresponding to (19), is

$$w = \exp\left(i\frac{k}{\lambda}\theta \cdot \xi\right) + A_{\lambda^{-2}q}\left(\theta', \theta, \frac{k}{\lambda}\right) \frac{\exp(i(k/\lambda)|\xi|)}{|\xi|} + o\left(\frac{1}{|\xi|}\right) \tag{20}$$

while (18) can be written as

$$u = \exp\left(i\frac{k}{\lambda}\theta \cdot \xi\right) + \lambda A_{q \circ \lambda}(\theta', \theta, k)|\xi|^{-1} \exp\left(i\frac{k}{\lambda}|\xi|\right) + o\left(\frac{1}{|\xi|}\right). \tag{21}$$

Compare (19), (20), and (21) to obtain

$$A_{\lambda^{-2}q}\left(\theta', \theta, \frac{k}{\lambda}\right) = \lambda A_{q \circ \lambda}(\theta', \theta, k). \tag{22}$$

Let $k/\lambda := \mu$, $\lambda^{-2}q = p$. Then (22) can be written as

$$A_p(\theta', \theta, \mu) = \lambda A_{\lambda^2 p \circ \lambda}(\theta', \theta, \lambda\mu). \tag{23}$$

This is formula (6) with $p = q$ and $\mu = k$.

(c) *Formula* (7). Consider

$$[\nabla_x^2 + k^2 - q(x-a)]u = 0 \qquad \text{in } \mathbb{R}^3 \tag{24}$$

$$u = \exp(ik\theta \cdot x) + A_{q \circ T_{-a}}(\theta', \theta, k)|x|^{-1} \exp(ik|x|) + o(|x|^{-1}). \tag{25}$$

Let $x = a = \xi$. Then (24) becomes

$$[\nabla_x^2 + k^2 - q(\xi)]\, w = 0, \tag{26}$$

and

$$w = \exp(ik\theta\cdot\xi) + A_q(\theta', \theta, k)\,\frac{\exp(ik|\xi|)}{|\xi|} + o\left(\frac{1}{|\xi|}\right) \tag{27}$$

while

$$\begin{aligned} u = {} & \exp(ik\theta\cdot\xi)\exp(ik\theta\cdot a) + A_{q\,\circ\,T_{-a}}(\theta', \theta, k)|\xi + a|^{-1} \\ & \times \exp(ik|\xi + a|) + o\left(\frac{1}{|\xi|}\right). \end{aligned} \tag{28}$$

Note that

$$\begin{aligned} |\xi + a| &= |\xi| + \xi^0\cdot a + O(|\xi|^{-1}) && \text{as} \quad \xi \to \infty, \\ \xi^0 &:= \frac{\xi}{|\xi|} \to x^0 = \theta' && \text{as} \quad |\xi| \to \infty. \end{aligned} \tag{29}$$

Thus (28) becomes

$$\begin{aligned} u = \exp(ik\theta\cdot a)\Bigg[& \exp(ik\theta\cdot\xi) + A_{q\,\circ\,T_{-a}}(\theta', \theta, k) \\ & \times \exp\{ik(\theta' - \theta)\cdot a\}\,\frac{\exp(ik|\xi|)}{|\xi|} + o\left(\frac{1}{|\xi|}\right)\Bigg]. \end{aligned} \tag{30}$$

Since u solves the homogeneous linear equation (24) the expression in brackets in (30) solves Eq. (26). Compare (27) and (30) to obtain

$$A_q(\theta', \theta, k) = A_{q\,\circ\,T_{-a}}(\theta', \theta, k)\exp\{ik(\theta' - \theta)\cdot a\}. \tag{31}$$

This is formula (7).

2. *Proof of Theorems* 2 *and* 3

(a) *Proof of Theorem* 2. The basic auxiliary result is the following uniqueness theorem from [3, 4].

PROPOSITION 1. *Let* $q_j \in Q_b$, $j = 1, 2$. *If* $A_{q_1}(\theta', \theta, k) = A_{q_2}(\theta', \theta, k)$ *at a fixed* $k > 0$ *and all* $\theta', \theta \in S^2$, *then* $q_1 = q_2$.

Theorem 2 follows immediately from formula (5) and Proposition 1.

Indeed, it is well known (and follows immediately from separation of variables) that if the potential is spherically symmetric

$$q(x) = q(|x|) = q(r), \qquad r = |x| \tag{32}$$

then (8) holds for all $k > 0$. Assume that (9) holds and (8) holds at a *fixed* $k > 0$ with $A = A_q$. Then

$$\begin{aligned} A_q(\theta', \theta, k) &= A_q(\theta' \cdot \theta, k) = A_q(R\theta' \cdot R\theta, k) \\ &= A_{q \circ R}(R\theta', R\theta, k), \qquad \forall R \in O(3). \end{aligned} \tag{33}$$

Here the second equation follows from (14) and the third equation is formula (5). Since $q \circ R \in Q_b$ if $q \in Q_b$, and since θ' and θ are arbitrary, one can write (33) as

$$A_q(\alpha', \alpha, k) = A_{q \circ R}(\alpha', \alpha, k), \qquad \forall R \in O(3), \forall \alpha', \alpha \in S^2, \tag{34}$$

where $\alpha' = R\theta'$, $\alpha = R\theta$. By Proposition 1, it follows that

$$q = q \circ R, \qquad \forall R \in O(3). \tag{35}$$

This is equivalent to (32). Theorem 2 is proved.

(b) *Proof of Theorem* 3. The basic auxiliary result is the following uniqueness theorem from [1, p. 85].

PROPOSITION 2. *Assume that* $A_{\Gamma_1, \zeta_1}(\theta', \theta, k) = A_{\Gamma_2, \zeta_2}(\theta', \theta, k)$ *for all* $\theta', \theta \in S^2$ *and a fixed* $k > 0$. *Then* $\Gamma_1 = \Gamma_2$ *and* $\zeta_1(s) = \zeta_2(s)$.

Note that if Γ is a sphere and $\zeta = \text{const}$, then (8) holds with $A(\theta', \theta, k) = A_{\Gamma, \zeta}(\theta', \theta, k)$ for all $\theta', \theta \in S^2$ and all $k > 0$. This follows from the analytical solution of the scattering problem by separation of variables. Assume now that (8) holds at a single fixed $k > 0$. Then, by (8) and (14),

$$A_{\Gamma, \zeta}(\theta', \theta, k) = A_{\Gamma, \zeta}(\theta' \cdot \theta, k) = A_{\Gamma, \zeta}(R\theta' \cdot R\theta, k) \tag{36}$$

and

$$A_{\Gamma, \zeta}(\theta', \theta, k) = A_{\Gamma \circ R, \zeta \circ R}(R\theta' \cdot R\theta, k) \tag{37}$$

by formula analogous to (5). Here $\Gamma \circ R$ is the surface Γ rotated by the element $R \in O(3)$, and $(\zeta \circ R)(s) = \zeta(R^{-1}s)$. From (36) and (37) it follows that

$$A_{\Gamma, \zeta}(\alpha', \alpha, k) = A_{\Gamma \circ R, \zeta \circ R}(\alpha', \alpha, k), \qquad \forall R \in O(3), \forall \alpha', \alpha \in S^2. \tag{38}$$

By Proposition 2, Eq. (38) implies

$$\Gamma = \Gamma \circ R, \qquad \zeta = \zeta \circ R, \qquad \forall R \in O(3). \tag{39}$$

Thus Γ is a sphere and $\zeta(s) = \zeta(|s|) = \text{const}$ on the sphere Γ. Theorem 3 is proved.

Remark 1. Consider compactly supported potentials $q = q(\rho, z)$, $z = x_3$, $\rho = (x_1^2 + x_2^2)^{1/2}$. Let R_φ, $0 \leqslant \varphi \leqslant 2\pi$, denote rotations about the x_3-axis. Then $q \circ R_\varphi = q$. By formula (5) for $R = R_\varphi$ one has

$$A_q(\theta', \theta, k) = A_{q \circ R_\varphi}(R_\varphi \theta', R_\varphi \theta, k) = A_q(R_\varphi \theta', R_\varphi \theta, k). \tag{40}$$

This symmetry property

$$A_q(\theta', \theta, k) = A_q(R_\varphi \theta', R_\varphi \theta, k) \tag{41}$$

is a necessary property of the scattering amplitude corresponding to the potential which is axially symmetric about the x_3-axis; i.e., $q(x) = q(\rho, z)$. If $q \in Q_b$, then (41) is also sufficient for $q(x)$ to be axially symmetric about the x_3-axis. This is proved as in Theorem 1.

Remark 2. Consider the equation

$$[\nabla^2 + k^2 + k^2 v(x)]\, u = 0 \qquad \text{in} \quad \mathbb{R}^3,\ k > 0, \tag{42}$$

for $v \in Q_b$. At a fixed $k > 0$ the scattering of a plane wave by the inhomogeneity $v(x)$ is identical with the scattering of this wave by the potential $q(x) = -k^2 v(x)$. Therefore, by Theorem 2, $v(x) = v(|x|)$ if and only if condition (8) holds at a fixed $k > 0$, where $A(\theta', \theta, k)$ is the scattering amplitude corresponding to Eq. (42).

Remark 3. The argument in this paper is based on uniqueness theorems. If $q \in Q^{(\beta)} := \{q\colon |q(x)| \leqslant c(1 + |x|)^{-\beta},\ q = \bar{q}\}$, $\beta > 3$, Proposition 2 is no longer valid. However, for this class of potentials the following *uniqueness theorem* holds: *if* $A_{q_1}(\theta', \theta, k) = A_{q_2}(\theta', \theta, k)$ *for all* $\theta', \theta \in S^2$ *and almost all* $k > 0$, *then* $q_1 = q_2$. The argument used in the proof of Theorem 2 shows that if $q \in Q^{(\beta)}$, $\beta > 3$, and if (8) holds for all $k > 0$, then $q(x) = q(|x|)$. The uniqueness theorem holds even for $q \in Q^{(\beta)}$ with $\beta > 1$ (see [9]). Therefore, the following theorem holds.

THEOREM 4. *If* $q \in Q^{(\beta)}$, $\beta > 1$, *then* $q(x) = q(r)$ *iff* (8) *holds for almost all* $k > 0$.

Remark 4. If $q \in Q^{(\beta)}$, $1 < \beta \leqslant 3$, then $A(\theta', \theta, k)$ is not necessarily continuous in θ' and θ so that (8) is understood as equality of kernels of operators on $L^2(S^2)$.

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